ECONOMIC ANALYSIS GROUP
DISCUSSION PAPER

SEQUENTIAL ENGLISH AUCTIONS: A THEORY OF OPENING-BID FISHING

by

Joseph Uri Podwol

EAG 15-1   March 2015

* U.S. Department of Justice, joseph.podwol@usdoj.gov.

Sequential English auctions: A theory of opening-bid fishing*

Joseph Uri Podwol

March 4, 2015

Abstract

Cassady (1967) describes an auction in which the auctioneer "fishes" for an opening bid, calling out lower and lower amounts until an opening bid is eventually placed. Once a bid is placed, it is not uncommon for the bidding to escalate above the initial starting price. The current study explains this puzzle in a model in which an auctioneer sells an indivisible good via English ascending-price auction and cannot commit to keeping the item off the market should the initial starting price fail to elicit any bids. A key insight of the paper is that the well-known strategy equivalence between the English auction and the second-price auction fails to extend to the sequential setting. This difference has important implications for the equilibrium starting-price path, giving rise to a *Coase conjecture* in the English auction but not in the second-price auction.

1 Introduction

In his oft-cited survey of all things auction, Ralph Cassady (1967, pages 57, 105, 113) describes auctions for a range of products from livestock to antiques in which the auctioneer *fishes* for an opening bid. The auctioneer begins by announcing an opening bid and goes on to solicit bids from the set of assembled buyers. If a bid is placed, the auction proceeds in typical English ascending-price fashion – soliciting higher and higher bids until the final amount is hammered down with a gavel. But should the auctioneer fail to find a bidder, the opening bid is reduced and a second attempt is made. This process continues until an opening bid ultimately elicits a response from a buyer. Cassady points out an astonishing outcome of this process: once a bid is placed, it is not uncommon for the bidding to progress beyond the amount of the initially proposed opening bid.

*Antitrust Division, U.S. Department of Justice, USA. joseph.podwol@usdoj.gov. The views expressed herein are the author's and do not necessarily reflect those of the U.S. Department of Justice. This paper is adapted from a chapter of my Ph.D. dissertation at Cornell University. Thanks to Talia Bar, Justin Johnson, Deborah Minehart, Henry Schneider, Michael Waldman and various seminar participants for valuable suggestions. All errors are my own.

More recent evidence suggests that this outcome extends to ascending-price auctions conducted online. In a quote culled from an eBay message board, this seller of homemade jewelry makes a similar discovery:

> *I have been in the practice of listing items for the least amount of money I would be willing to sell them for. I discovered that another seller was selling items similar to mine and starting them at 99 cents [...] I noticed that many of her bids went way above what my similar items started at, where many of my items started at the higher price just stagnated without bids. – Treasures_by_Cynthia*

It is surprising, given the breadth of the literature on optimal auctions, that the tactic of opening bid fishing has not received more attention. That the high bid is negatively correlated with the starting price (or opening bid) may well be explained by *affiliated values* or by *auction fever*. In the affiliated values model of Milgrom and Weber (1982), buyers shade their bids in order to avoid the winner's curse. A lower starting price allows for more bids, which reveal information to buyers and leads to less bid shading. Ockenfels, Reiley, and Sadrieh (2007) define auction fever as "an excited and competitive state-of-mind, in wich the thrill of competing against other bidders increases a bidder's willingness to pay in an auction." This definition encompasses a variety of cognitive biases which offer a common prediction that a lower starting price leads to increased competition, which serves as a trigger for excitement and consequently higher bids.[1] While models of affiliated values and auction fever offer compelling insights, neither explains why the seller wouldn't simply set a low initial opening bid, as opposed to fishing for one.

To rationalize opening bid fishing and the concurrent *start low end high* phenomenon, we analyze a dynamic model in which each period offers the seller an opportunity to sell a single item via English auction with an announced starting price. Buyers' valuations are independently and identically distributed and the lowest possible bidder type exceeds the seller's valuation. The seller cannot commit in any period to a take-it-or-leave-it offer. Therefore, if the auction fails to elicit a bid, she relists in the following period and continues to conduct auctions successively as long as it is sequentially rational to do so. Anticipating this behavior, buyers may wish to hold off bidding in the current auction and instead wait for a subsequent auction where the item may be obtained at a lower price. For instance, a buyer with a valuation of S, say, gains nothing by winning an auction started at S. But should the item go unsold in the intial auction, his expected surplus in the following period is positive since the starting price will be reduced. In equilibrium, a buyer whose valuation lies between S and some threshold value holds off bidding in the auction with a starting price of S, but bids up to his valuation when the item is relisted with a lower starting price.

[1] Ku, Galinsky, and Murninghan (2006) study a similar mechanism in which auction prices are negatively correlated with starting prices, which they attribute to the tendency for: i) bidders who enter the bidding early to justify the sunk cost of participation and ii) bidders who arrive late to infer greater value to auctions with more bids.

The preceding argument explains the outcome described by Cassady that after the initial starting price fails to induce any bids, the item may later be hammered down at a price greater than the initial starting price. This would be the case if the two highest valuations among the assembled buyers were both greater than the initial starting price but below the threshold value. This is actually the most likely outcome when the time between auctions is sufficiently short. In such circumstances, waiting for the subsequent auction is virtually costless to buyers. Consequently, the seller must set the initial starting price low in order to induce buyers with high valuations (should there be any present) to bid. Despite the low initial starting price, there will still be buyers that prefer to wait for the subsequent auction–where the starting price will be even lower–before bidding. Eventually, the seller reduces the starting price low enough to induce all buyers to bid. The high bid (i.e. the amount paid by the high bidder) in this auction will be equal to the second-highest bidder's valuation, which is greater than the low initial starting price.

This research contributes to the budding literature on credible sales mechanisms. McAdams and Schwartz (2007) and Vartiainen (2013) are recent studies which model environments in which the seller, having announced her intention to sell an item using a particular auction mechanism, cannot keep from deviating when information revealed during the auction makes it profitable to do so.[2] The most closely related research is McAfee and Vincent (1997), who model a seller that cannot commit to a reserve-price policy when offering an item for sale via first-price or second-price sealed-bid auction, respectively.[3] They show that under both mechanisms, the reserve-price path is declining similar to the declining starting-price path in the current study. The current study shows, however, that Cassady's start low end high phenomenon arises when the item is offered via English auction but not when offered via first- or second-price auction. A seller that offers the item via English auction follows a starting price path that is lower than the corresponding reserve price path of a seller that uses only first- or second-price auctions, respectively. In the limit, as the time between successive English auctions becomes short, the initial starting price convergest to the lowest possible buyer valuation. A related result in the sequential bargaining literature is the "Coase conjecture," which says that when a seller cannot commit to a take-it-or-leave-it offer, her initial offer converges to the lowest buyer valuation type as the time between successive offers becomes short.[4] The current study is the first to extend this result to a setting with multiple buyers.

Skreta (2013) studies a similar model to McAfee and Vincent (1997) but where

[2]Along these lines, Bester and Strausz (2001) model a principle-agent relationship where the principle cannot commit to following the rules of the mechanism.

[3]A "reserve price" in a sealed-bid auction indicates the lowest allowable bid. This acts similar to a "starting price" in an open-outry or English auction.

[4]Coase (1972) proposed that a durable-good monopolist that cannot commit to not make additional sales in subsequent periods must reduce the price in the initial period to marginal cost. Gul, Sonnenschein, and Wilson (1986) formalized the argument and pointed out that the model is mathematically equivalent to one in which a monopoly seller bargains with a buyer whose valuation is known only to himself and the the seller is unable to commit to a take-it-or-leave-it offer.

the seller can commit, after a fixed number of periods, to keeping the item off the market.[5] Burguet and Sakovics (1996) consider a sequential first-price auction model where the number of bidders is endogenous. In both instances, the starting price path is quite different from the one studied here.

This research speaks to the literature on auction theory and mechanism design. Vickrey (1961) first demonstrated that the English and second-price auctions are strategically equivalent and that they yield the same expected revenue to the seller.[6] Subsequent research showed that in more complex informational environments, they are not strategically equivalent nor are they revenue equivalent.[7] In the setup considered here, the English and second-price auctions are not strategically equivalent yet they are revenue equivalent. The English auction conveys information to buyers that once a bid is placed, the current auction will be the last. This information is not available to buyers in a second-price auction where bids are sealed until the winner is announced. The additional information revealed on in the English auction induces buyers to bid in the current period when they otherwise would have waited for a later period. Anticipating this response from competing bidders, each bidder becomes more passive in deciding whether to be the first to bid. This in turn forces the seller to reduce the starting price relative to the reserve price in a second-price auction in order to induce buyers to be the first to bid. These effects turn out to be offsetting so that the expected revenue is the same whether the seller uses English or second-price auctions.

The paper is organized as follows. Section 2 describes the model. Section 3 characterizes the equilibrium of the sequential English auction game and proves the Coase conjecture. Section 4 demonstrates how the start low end high phenomenon arises from equilibrium behavior. Section 5 reconciles our results with those of McAfee and Vincent (1997) and Section 6 concludes.

2 The model

Consider an auction marketplace consisting of a single seller and $n \geq 1$ potential buyers indexed $i = 1, 2, ..., n$. The seller has one unit of a particular item to sell. Buyers are risk neutral, have unit demands, and differ only in their valuation of the item. Each buyer's valuation, denoted v, is known only to himself. Valuations are assumed to be independent and identically distributed according to F, a continuous distribution with density f, bounded between zero and infinity, over the support $[\underline{v}, \bar{v}]$, where $\underline{v} > 0$. The seller's valuation is normalized to zero and the seller is risk neutral.[8] Assume the seller's valuation, the distribution of buyer valuations F, and n are common knowledge.

[5] The primary contribution of Skreta (2013) is endogenizing the seller's choice of mechanism.

[6] They are also revenue equivalent to other common mechanisms such as the first-price and Dutch auctions. See Meyerson (1981) and Riley and Samuelson (1981).

[7] Milgrom and Weber (1982) and Caillaud and Mezzetti (2004) are two examples.

[8] The seller's valuation need not be zero, but it is important for the analysis that it be strictly less than $\underline{v}$. This is known as the "gap" case in the literature. The significance of this assumption is

The time horizon is infinite with periods indexed $t = 1, 2, ...$ The pool of buyers is fixed over time as are their valuations. In period 1, the seller conducts an English auction and announces an opening bid or "starting price," s_1. The English auction proceeds as follows: Beginning with s_1, the auctioneer calls for bids. If a bid is placed, the seller pauses to invite additional bidders to enter. Each buyer indicates his active status by pressing a button. Once all the buyers that wish to enter have done so, the price increases continuously. Bidders may drop out at any point by releasing their button. The price continues to increase until all but one bidder has dropped out. At that point, the one remaining bidder is awarded the item and pays the "high bid," equal to the price at which the last bidder dropped out, or s_1 if no other bidders have entered. If no bids are placed at s_1, the seller may conduct an English auction in period 2. If the auction item fails to elicit a bid in period 2, the game continues on to period 3 and so on. As the game advances from one period to the next, all buyers and the seller discount returns accrued in the following period by factor $\delta \in (0, 1)$, which is common knowledge.

This setup models open-bid fishing as a repeated game, where the sales mechanism in each period is an English auction. The English auction modeled here differs from the typical "button" auction introduced by Milgrom and Weber (1982) as a stylized approximation to the English auction. In the button auction, the seller announces a starting price and buyers decide simultaneously whether to enter, which they indicate by pressing a button. The price then increases continuously and buyers may exit by releasing their button. The decision to exit is irrevocable. The English auction modeled here adds a bit of realism in that in the English open-outcry auctions often seen in practice, buyers who were not active at the beginning can submit bids later on. In fact, the auctioneer generally cannot discern which bidders are active and which are not until all but one have dropped out. Here, we allow the entry decision to take place over possibly two stages. In the first stage, buyers simultaneously decide whether to press their buttons. If at least one buyer presses his button in stage 1, then a stage 2 is entered, whereby all buyers are made aware that a bid has been placed in stage 1, and all those who did not bid in stage 1, may do so in stage 2. Following stage 2, the price rises continuously as in Milgrom and Weber (1982) until all but one bidder has dropped out. If no bids are placed in stage 1, then stage 2 is foregone and the game moves immediately to the next period, where the seller may announce a new starting price.

The sequential nature of the game arises because the seller cannot – as is typically assumed in the literature on optimal auctions – commit to keeping the item off the market should the initial auction fail to elicit any bids. Absent commitment power, the seller continues to relist in subsequent periods since selling the item to even the lowest valuation buyer is always preferred to keeping it. It should be noted that due to the assumption that $\underline{v} > 0$, all equilibria of the game are necessarily stationary. This precludes the existence of *reputational equilibria* (Ausubel and Deneckere 1989), in which the seller reduces the starting price at an arbitrarily slow rate.

discussed later in this section.

Though the seller's objective function need not be concave to insure a stationary equilibrium,[9] such considerations complicate the analysis without adding to the economic substance. Therefore, we impose the following regularity condition.

Assumption 1 *For any $u, v \in [\underline{v}, \bar{v}]$, $v - \frac{F(u)-F(v)}{f(v)}$ is strictly increasing in v.*

The sequential English auction game is shown to have multiple equilibria. All of the equilibria have a declining starting-price path and have the same expected revenue and the same probability of sale in each period. The equilibria differ only in the starting price used in each period. We employ a selection criteron to settle on an equilibrium of the game in which rationalizes the start low end high phenomenon described by Cassady (1967).

3 Equilibrium characterization

The equilibrium concept is *perfect-Bayesian equilibrium* (PBE). Any equilibrium is a history-contingent sequence of the seller's starting prices, s_t, buyers' bidding decisions, and updated beliefs about the valuations of existing buyers satisfying the typical consistency conditions. Formally, let $H_\tau = \{s_1, s_2, ...s_\tau\}$ denote the history through period τ of a game that has not ended prior to period τ. Since a bid placed in any period $t < \tau$ necessarily results in the game ending in that period, H_τ consists only of the seller's starting prices with the implicit assertion that no bids have been placed to that point. A strategy for the seller in period t is a starting price which maximizes expected discounted revenue given her beliefs over buyer valuations and given equilibrium behavior in what follows. A strategy for each buyer in period t involves two decisions – whether to bid in the period-t auction and if so, what price to drop out at – which jointly maximize expected discounted surplus given his beliefs over valuations and given equilibrium behavior. We restrict attention to monotonic bidding strategies, which is necessary for an equilibrium in symmetric strategies.

Whether or not the game proceeds to the next period relies on a buyer's decision of whether to bid at the starting price, that is when it is not known if other buyers intend to bid in that period. When a buyer does ultimately place a bid, it insures that the item will sell and the game will end at the conclusion of the bidding. An important distinction then is between *initial bidders*, those willing to bid at the starting price, and *interim bidders*, those who bid only after the bidding has begun. Note that there can be more than one initial bidder since the distinction is a counter-factual, determined by what the buyer *would do* if no other buyers had submitted a bid.

Lemma 1 *The following characterize the equilibrium in any PBE:*

1. *It is weakly dominant for an initial bidder to drop out when the price equals his valuation.*

[9] See McAfee and Vincent (1997).

2. *In any period t, there exists a marginal type β_t, such that every buyer whose valuation exceeds β_t, bids at the starting price.*

3. *Regardless of the history, all buyer types bid at the starting price when the starting price is at or below $\underline{v}$.*

4. *There exists a period $T < \infty$, endogenously determined, such that the game ends in at most T periods.*

All proofs are in the Appendix. Result 1 of the lemma is a standard result in auction theory but applied to the sequential setting. Dropping out at one's valuation is unique among all symmetric decision rules and so we assume initial bidders follow that in what follows. The bidding of interim bidders is as yet undetermined; this will be pinned down by the equilibrium refinement.

Result 2 of the lemma is the *successive skimming* property, which is common to the dual literatures on sequential bargaining and durable good monopoly. Analogous to those models, a buyer bids at a given starting price only if the payoff from bidding in the current period is sufficiently large to have him forgoe future opportunities. Since any buyer whose valuation exceeds β_t necessarily bids in period t, it follows that if the item remains unsold after period t, it must be that all valuations are below β_t. This makes for a simple updating rule in which β_t becomes the highest type in period $t+1$, denoted u_{t+1}. In what follows, we refer to u_τ as the *state* in period τ and β_τ as the *screening level* in period τ, both of which are standard usage in the aforementioned literature.

A second implication of Result 2 is that initial bidders have higher valuations than interim bidders. Since β_t determines a minimum valuation type, any interim bidder in period t must not have had a valuation above β_t, otherwise he would have been an initial bidder. Thus the allocation of the item amongst buyers will be unaffected by the strategies of interim bidders since the item can only be obtained by the buyer with the highest valuation. However, the bidding of interim bidders raises the price above what it would have been absent such bidding, taking everything else as given. To see this, consider the bidding decision of an interim bidder after an initial bid has been placed. The fact that he was not an initial bidder indicates to him that he will ultimately lose the bidding and so has no reason to bid. On the other hand, he has no reason not to bid as doing so is costless. Therefore, any strategy that assigns positive probability to any amount up to and including the buyer's own valuation is individually rational. Only bids above the buyer's own valuation are ruled out in equilibrium.

To understand how interim bidders affect the payoffs of initial bidders, consider some period t and suppose that there exists only one initial bidder. If interim bidders were precluded from bidding, the lone initial bidder would be the only buyer to submit a bid and he would pay a price equal to the starting price, s_t. This is the outcome in the analogous second-price auction, where there are no interim bidders. If instead, interim bidders are allowed to bid, the price paid by the initial bidder would be determined by the highest bid placed among all interim bidders and would only equal s_t if all initial bidders declined to bid.

Let ρ_t denote the expected price paid by a lone initial bidder, taking into account the (possibly mixed) strategy followed by interim bidders. Without applying an equilibrium refinement criterion, ρ_t is indeterminate. We assume in what follows that $\rho_t(s_t, \beta_t)$ is increasing in both arguments and strictly so for the first argument.[10]

3.1 Characteristics of all equilibria

That the screening level eventually reaches $\underline{v}$ (Result 4 of the Lemma), whereby the game necessarily ends (Result 3), implies that the equilibrium can be derived via backward induction. A complication arises because the number of periods required for the starting price to reach $\underline{v}$ is determined endogenously, so the number of periods to be inducted upon must also be solved for. The details are presented in Appendix A.2.

Proposition 1 *A PBE consists of a sequence of screening levels $\{\beta_t\}_{t=1}^{T}$ and corresponding starting prices $\{s_t\}_{t=1}^{T}$ such that:*

1. *In any period t, the seller chooses screening level $\beta_t = \beta(u_t)$ to satisfy*

$$\beta_t f(\beta_t) + F(\beta_t) - F(u_t) \leq 0, \tag{1}$$

with a strict equality when $\beta_t > \underline{v}$. The choice of β_t depends only on u_t and the density f, and is independent of δ, n, and ρ_t.

2. *To induce a screening level of β_t, given ρ_t, the period-t starting price $s_t = \sigma(\beta_t)$ satisfies the following sequential rationality condition,*

$$[\beta_t - \rho_t(\sigma(\beta_t), \beta_t)] F_{Y_1}(\beta_t) = \delta \left([\beta_{t+1} - \rho_{t+1}] F_{Y_1}(\beta_{t+1}) + \int_{\beta_{t+1}}^{x} F_{Y_1}(Y_1)\, dY_1 \right), \tag{2}$$

where $\beta_{t+1} = \beta(\beta_t)$ and $\rho_{t+1} = \rho_{t+1}(\sigma(\beta_{t+1}), \beta_{t+1})$.

3. *The allocation of the item, the seller's revenue, and the probability of sale in a given period are independent of ρ_t.*

Equation (1) is the solution to the seller's problem, taking into account the equilibrium strategies of buyers and the optimal screening levels in subsequent periods. In a static version of the game, this same condition solves the seller's problem in selecting the optimal starting price, where $\overline{v}$ would take the place of u_t. However, in a static setting, the screening level and starting price are one and the same. Within

[10] This assumption is satisfied trivially when interim bidders opt not to bid, in which case $\rho_t(s_t, \beta_t) = s_t$, and will be shown to be true when they bid up to their valuations as in the equilibrium selected under our refinement.

the sequential setting, sequential rationality requires that the seller set the starting price below the screening level in order to induce bids from the desired set of buyer types.

Equation (2) indicates that the seller's choice of starting price makes a type β_t buyer indifferent between bidding at the starting price in period t and waiting one more period, taking into account equilibrium behavior. The left-hand side of the expression indicates a type β_t buyer's expected surplus from bidding at the starting price, $\sigma(\beta_t)$, when he is the lowest type to do so. His expected payment would then be given by $\rho_t(\sigma(\beta_t), \beta_t)$. $F_{Y_1}(\beta_t)$ denotes the probability that type β_t is the lone initial bidder.[11] The right-hand side of equation (2) indicates a type β_t buyer's expected continuation surplus were he not to bid, given an equilibrium starting price of σ_{t+1} and screening level of β_{t+1} in the period to follow. In the following period, since $\beta_t > \beta_{t+1}$, he receives an amount given by the first term in the event he is the lone initial bidder, and an amount given by the second term when he is bidding against at least one other initial bidder.[12]

Result 3 of the proposition, that the seller's revenue is independent of ρ_t, would seem counter-intuitive since a higher value of ρ_t translates to a larger expected payment for the winning bidder, holding fixed the seller's starting price. However, the seller takes ρ_t into account when selecting the starting price. The higher is ρ_t, the lower the starting price must be in order to induce bids from buyers whose valuations are at least β_t, the desired screening level.[13] In this way, the equilibrium starting prices corresponding to every possible value of ρ_t lie along a continuum. The equilibrium we will select for using our refinement criterion yields the highest possible ρ_t and hence lowest possible starting price. The equilibrium yielding the highest possible starting price is the case in which $\rho_t = s_t$, which is the payment the lone bidder would make in the analogous second-price auction. One implication is that the starting price path in any equilibrium of the sequential English auction game is lower period by period (strictly so for $\rho_t > s_t$) than the equilibrium starting price path in the sequential second-price auction game. A second implication is that all equilibria of the English auction game are revenue-equivalent to the equilibrium of the second-price auction game.

3.2 Equilibrium refinement

Having characterized equilibrium behavior of sellers and initial bidders under any equilibrium strategy played by interim bidders, we now apply a refinement to select the equilibrium in which interim bidders are most aggressive. This equilibrium is important as it gives rise to the Coase conjecture.

Consider a perturbed version of the game in which, in a given auction, every

[11] Using conventional notation, let Y_1 denote the maximum of the $n-1$ competing buyers' valuations. As the v_i are independent, $F_{Y_1}(\beta_t) \equiv F(\beta_t)^{n-1}$.

[12] All expectations are taken conditional on state u_t being reached. Expression (2) does not explicitly include u_t due to a canceling of terms.

[13] This follows from equation (2) along with the assumption that ρ_t is strictly increasing in s_t.

possible move is played by each buyer with positive probability.[14] Each buyer bids at the starting price, s_t, with some probability strictly between zero and one and each buyer drops out at any price in $(s_t, \bar{v}]$ with some probability strictly between zero and one. The limits of equilibria of such perturbed games as the tremble probabilities go to zero are *extensive-form trembling-hand-perfect equilibria* (ETE).[15]

Proposition 2 *There exists a unique ETE in which each interim bidder bids up to his valuation with probability 1.*

The logic behind the proposition is that trembles make it so that an interim bidder may with positive probability win the auction at a price below his valuation. This is because, in contrast to equilibrium behavior, a buyer who bids at the starting price before it is known whether others will bid may drop out at a price less than the interim bidder's valuation. It is then in the interim bidder's interest to remain active until his valuation is reached. The same conclusion could also be reached less formally by appealing to the seller's incentive to entice interim bidders to remain active. It is costless for interim bidders to continue bidding up to their valuations, while the seller strictly prefers that they do so. Thus the equililbria in which interim bidders do not stay active up to their valuations are not robust to the possibility that the seller can pay buyers to remain active. We assume in all further discussion of the English auction that the selected equilibrium is the one that is played. The expected payment made by a lone initial bidder is then,

$$\begin{aligned} \rho(s_t, \beta_t) &= E\left[\max\{s_t, Y_1\} | Y_1 < \beta_t\right] \\ &= \beta_t - \frac{\int_{s_t}^{\beta_t} F_{Y_1}(Y_1) dY_1}{F_{Y_1}(\beta_t)}. \end{aligned} \tag{3}$$

The aggressive bidding by interim bidders in the ETE puts downward pressure on the seller's starting price. In the limit as δ goes to unity, this gives rise to the following *Coase conjecture.*

Proposition 3 *In the ETE, for every $\varepsilon > 0$, there exists a $\bar{\delta} < 1$ such that for all $\delta \geq \bar{\delta}$, and for any initial screening level, $\beta_1 = \beta(\bar{v}) \in [\underline{v}, \bar{v}]$, the seller's initial starting price, $\sigma(\beta_1)$, is less than $\underline{v}+\varepsilon$.*

The intuition for this result is fairly straightforward. If all buyers believe that in the event they are the lone initial bidder, the price they will pay will be driven up by

[14] Trembles by the seller are uninteresting, since the starting price is known to all buyers before the auction begins.

[15] The ETE is an extension, due to Selten (1983), of the *trembling-hand-perfect* equilibrium concept of Selten (1975). A trembling-hand perfect equilibrium is one that takes the possibility of off-the-equilibrium play into account by assuming that the players, through a tremble, may choose unintended strategies, albeit with negligible probability. When extending this concept to extensive-form games, the modeler may choose to interpret a tremble as a mistake in a player's choice of action at a particular information set or as a mistake in a player's entire strategy choice. The ETE concept employs the former interpretation.

interim bidders, they will be more hesitant to place a bid at the starting price when it is not known if any other buyers will do so. It may make more sense to wait for a subsequent period, where the starting price will be reduced, before bidding. When the time between periods is very short (i.e. δ is close to 1), there is virtually no cost to waiting until the terminal period where starting price is reduced to the lowest valuation type. For the seller to induce the marginal type to bid in period 1, she must guarantee him the surplus he would have obtained in the terminal period. The only way to do this is to drop the starting price in period 1 to the lowest valuation type. For reasons that I discuss in Section 5, this result does not extend to values of ρ_t less than the value given by expression (3).

4 Bidding dynamics

The Introduction described a sequence of English auctions which give rise to a counter-intuitive result. Cassady (1967) explains that after failing to elicit any bids in the initial auction, the seller lowers the starting price whereupon the bidding escalates beyond the amount of the initial starting price. The current section seeks to rationalize this *start low end high phenomenon.* The first result – the "weak gap" property – demonstrates that this is a possible outcome. The second result– the "strong gap" property – demonstrates that when the time between auctions is sufficiently short, the subsequent auction is actually more likely to end with a price at least as high as the initial starting price than is the initial auction.

We begin by motivating the general results with a simplifying example. Let $n = 2$ and suppose that buyer valuations are drawn from a Uniform $[a, 1 + a]$ distribution, $a \in (0, 1)$. Within the PBE, the subgame beginning at some arbitrary period t is characterized by a state variable, u, which denotes the highest possible valuation among the contingent of buyers, given that the item is still available to that point. This value is obviously equal to $1+a$ in period 1 and decreases from there. The PBE is stationary so u does not depend explicitly on t. The seller's period-t problem reduces to one of choosing the optimal screening level in the current period, $\beta(u)$, subject to the constraint that the type-β buyer is indifferent between bidding in period t and bidding in period $t + 1$ and given sequential rationality in what follows. Inserting $F(x) = x - a$ into (1), the solution to the seller's problem is given by,

$$\beta(u) = \begin{cases} u/2 & \text{if } u \geq 2a \\ a & \text{otherwise} \end{cases} . \tag{4}$$

At any state u, the seller cuts the demand curve in half, serving the top half, until u becomes sufficiently small so that $u/2$ falls below a, whereby the seller reduces the starting price to a which induces all buyer types to bid.

Examination of (4) shows that the length of the game (i.e. the maximum number of periods until a sale is made) is determined by the value of a. For any integer, k, the game will last at most k periods for $a \in \left[\frac{1}{2^k-1}, \frac{1}{2^{k-1}-1}\right)$. In what follows, we solve a two-period game and then show how those results extent to a game of any finite length.

4.1 A linear two-period game

Let $a \in [1/3, 1)$ so that the game ends in at most two periods. The sequence of screening levels is $\{\beta_1, \beta_2\} = \left\{\frac{1+a}{2}, a\right\}$ following from (4). To induce all buyer types to bid in period 2, the seller must set a non-binding starting price of $s_2 = a$.[16] All buyers bid because there is no chance that the starting price will be reduced in subsequent periods.

To induce buyers whose valuations are at least β_1 to bid in period 1, the seller must set s_1 low enough that these buyer types strictly prefer bidding in period 1. Solving for s_1 requires making the type-$\beta_1$buyer indifferent between bidding in period 1 when he does not know if other buyers will bid in period 1 and waiting for period 2 as shown in (2). Since all buyer types bid in period 2, $\beta_2 = \rho_2 = a$. Subsituting $\beta_2 = \rho_2 = a$, $F_{Y_1}(y) = y - a$ and $\beta_1 = (1 + a)/2$ into (2) and solving for s_1 yields,

$$s_1 = a + \frac{1-a}{2}\sqrt{1-\delta}. \tag{5}$$

Notice that as δ goes to 1, s_1 converges to a as guaranteed by the Coase conjecture.

The *start low end high* phenomenon is explained within the model as follows. Conditional on period-2 being reached, the high bid in period 2 exceeds s_1 if there are at least two buyers whose valuations exceed s_1. The fact that the period-1 auction failed to elicit any bids implies that there are no buyers whose valuation is as high as β_1. However, since $\beta_1 > s_1$, the high bid in period-2 exceeds s_1 with positive probability after conditioning on the fact that the period-1 auction failed to receive any bids. We refer to this as the "weak gap" property, as it is the gap between β_1 and s_1 that makes this result possible.

A somewhat surprising stronger result is that conditional on period 2 being reached, the period-2 is actually *more likely* to have a high bid exceeding s_1 than the period-1 auction when the time between relistings is sufficiently short. Let p_t denote the high bid in period t. This is equal to: the price at which all but one bidder has dropped out if two or more bidders entered the bidding in period t; or the starting price otherwise.[17] Let η_t be an indicator of whether the period-t auction elicits a bid (equal to unity if it does, zero otherwise). Further, let X_1 and X_2 denote the realizations of the highest and second-highest buyer valuations, respectively. The probability that the period-2 high bid exceeds s_1 is given by,

$$\begin{aligned} \Pr\{p_2 > s_1 | \eta_1 = 0\} &= \Pr\{X_2 > s_1 | X_1 < \beta_1\} \\ &= \frac{\beta_1 - s_1}{\beta_1 - a}. \end{aligned} \tag{6}$$

[16] Any starting price strictly less than a would also work except in the case of $n = 1$.

[17] The thought experiments that follow are concerned with whether p_t is greater than s_t or not. Therefore, it is immaterial whether p_t is assigned a value of s_t or zero if no bidders entered in period t.

Recall that as δ goes to 1, s_1 goes to a, so that the right-hand side of (6) goes to 1.

In period 1, the high bid exceeds s_1 if there is at least one initial bidder, whose valuation is at least β_1, and a second bidder who could be an initial bidder or an interim bidder but whose valuation exceeds s_1. The probability of this occuring is,

$$\begin{aligned} \Pr\{p_1 > s_1\} &= \Pr\{X_1 \geq \beta_1, X_2 > s_1\} \\ &< \Pr\{X_2 > s_1\} \\ &= [1-(s_1-a)]^2 \end{aligned} \tag{7}$$

where the inequality in the second line follows from the fact that β_1 is bounded above s_1. As δ goes to 1, the right-hand side of (7) converges to 1 from below. It follows that for δ sufficiently close to 1, $G \equiv \Pr\{p_2 > s_1 | \eta_1 = 0\} - \Pr\{p_1 > s_1\}$ is positive. We refer to this result as the "strong gap" property as it requires β_1 to be sufficiently larger than s_1.[18] The strong gap property is illustrated in Figure 1.1, which plots G over δ. The lower curves corresond to higher values of a. It is evident that each curve crosses the horizontal axis for δ sufficiently close to 1.

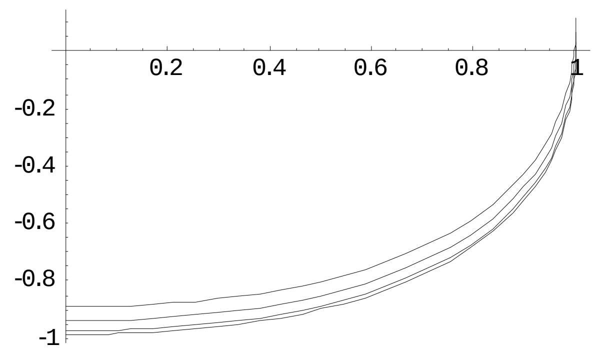

Figure 1.1: G as a function of δ for $a \in \{1/3, 1/2, 2/3, 3/4\}$.

4.2 A linear k-period game

The above result can be extended to a game of any finite length. As in the two-period case, the screening level in the last period of the game is $\beta_k^{(k)} = a$, so that all buyers are induced to bid. From (4), the state variable in the last period of a $k-$period game is

$$\beta_{k-1}^{(k)} = \frac{1+a}{2^{k-1}},$$

so that conditional on period k being reached, all buyers' valuations will lie in the interval $\left[a, \beta_{k-1}^{(k)}\right]$.[19] The Coase conjecture guarantees that the initial starting price

[18] Since β_1 is independent of δ and $s_1 = \sigma(\beta_1)$ is decreasing in δ, $\beta_1 - s_1$ is increasing in δ.

[19] That $\beta_{k-1}^{(k)} > a$ follows from the fact that for the game to end in at most k periods, it must be that $a \in \left[\frac{1}{2^k-1}, \frac{1}{2^{k-1}-1}\right)$.

converges to the minimum valuation type, a, as the time to relisting becomes sufficiently short. Therefore, conditional on period k being reached, the high bid in period k will necessarily exceed the initial starting price. In contrast, the high bid in the intial period exceeds the initial starting price only if there is one buyer whose valuation exceeds the initial screening level, $\beta_1^{(k)} = \frac{1+a}{2}$. Since $\beta_1^{(k)} > a$, there may be no buyers whose valuations are high enough to justify bidding in the initial period. Therefore, the probability that a high bid of at least s_1 is obtained in the period-k auction, conditional on period k being reached, is greater than the probability that a high bid of at least s_1 is obtained in period 1.

4.3 General results

I now establish in more general terms the weak and strong gap properties for the general model. The weak gap property shows that if the auction in some arbitrary period t fails to receive any bids, then some subsequent auction may end with a high bid strictly greater than the period-t starting price with positive probability.

Proposition 4 *Whenever $T > 1$, there exists a $\bar{\tau} > 1$ such that for any $\tau \leq \bar{\tau}$,*

$$\Pr\left\{p_{t+\tau} > s_t | \eta_{t+\tau-1} = 0\right\} > 0.$$

This says that if we consider any period $t < T$ in which the period-t auction fails to elicit a bid, any subsequent period's auction, say period $t + \tau$, may have a high bid of at least s_t as long as $\beta_{t+\tau-1} > s_t$. We know that this is necessarily true for $\tau = 1$ since the marginal type in a given period will always strictly exceed the starting price whenever there is at least one more period remaining in which the item could be obtained at a lower price. $\bar{\tau}$ is the highest integer value for which this continues to be true.

The strong gap property shows that as the time between auctions becomes sufficiently short, the auction in the terminal period is more likely to end at a price strictly greater than the period-t starting price than is the period-t auction itself.

Proposition 5 *Whenever $T > 1$, there exists some $\tilde{\delta} < 1$ such that for any $\delta \geq \tilde{\delta}$, and any $t < T$,*

$$\Pr\left\{p_T > s_t | \eta_{T-1} = 0\right\} > \Pr\left\{p_t > s_t\right\}$$

regardless of $T - t$, the number of periods required for the starting price to reach $\underline{v}$.

Proposition 5 says that the start low end high phenomenon is the most likely outcome when the auctioneer lowers the start price in quick succession (i.e. when δ is close to 1). This is because when doing so, the initial starting price (and all subsequent starting prices) become quite small. As such, upon failing to sell in the initial auction, and the $T - 2$ subsequent auctions, the auction in period T will necessarily have a high bid of at least s_1. This logic continues to hold when the period-1 starting price is replaced with period$-t$ starting price for any $t < T$.

5 English versus second-price auctions

We now return to the case where ρ_t can take on different values and discuss the differences in outcomes between English and second-price auctions when the seller lacks the ability to commit. Recall that the equilibrium in which $\rho_t = s_t$ generates the same outcomes (e.g., the allocation of the item, the probability of sale and the seller's revenue) as the second-price auction. This is the equilibrium in which interim bidders do not bid. In contrast, within the ETE of the English auction, interim bidders bid up to their valuations so that ρ_t is given by (3). We now consider differences in outcomes when between equilibria the equilibrium when ρ_t is given by (3) (the English auction) and when $\rho_t = s_t$ (the second-price auction).

The first difference to notice is that the Coase conjecture (Proposition 3) does not extend to the second-price auction.[20] Theorem 3 in McAfee and Vincent (1997) shows that the seller's profits converge to that of a one-shot auction with no reserve price in the limit as $\delta \to 1$, but even in the limit, the seller's initial starting price remains bounded above the minimum valuation type. Conditional on the sequence of screening levels, the seller's initial starting price (or reserve price in the case of the second-price auction) must be such that it makes the marginal bidder in period 1 indifferent between bidding in the current auction and waiting for the following period. When the time between auctions is very short, the seller has to guarantee the marginal bidder the same expected surplus that he would get in the terminal period, when the starting price is reduced to $\underline{v}$. Formally, the sequential rationality constraint reduces to:

$$\lim_{\delta \to 1} (\beta_1 - \rho_1) F_{Y_1}(\beta_1) = \int_{\underline{v}}^{\beta_1} F_{Y_1}(Y_1)\, dY_1, \tag{8}$$

where ρ_1 is still fully general as in equation (2) and Y_1 denotes the highest valuation of all buyers conditional on the marginal bidder being the highest valuation type among the n bidders. If $\rho_1 = \sigma(\beta_1)$ as in a second-price auction, then (8) becomes,

$$\lim_{\delta \to 1} [\beta_1 - \sigma(\beta_1)] F_{Y_1}(\beta_1) = \int_{\underline{v}}^{\beta_1} F_{Y_1}(Y_1)\, dY_1.$$

For this equality to hold in the limit, $\sigma(\beta_1)$ must strictly exceed $\underline{v}$ for any $\beta_1 > \underline{v}$.[21] The idea is that in the second-price auction, the seller screens out types in the interval $[\sigma(\beta_1), \beta_1)$ with a binding reserve price of $\sigma(\beta_1)$. Thus the marginal buyer pays $\sigma(\beta_1)$ for realizations of Y_1 in $[\sigma(\beta_1), \beta_1)$. By choosing $\sigma(\beta_1)$ small enough (but still greater than $\underline{v}$), the seller can make bidding in the initial period just profitable for marginal buyer.

[20] Further, it does not extend to any equilibrium other than the ETE.

[21] The right-hand side of the equality is the area under the F_{Y_1} curve over the domain $[\underline{v}, \beta_1]$. The left-hand side of the equality is a rectangle with height $F_{Y_1}(\beta_1)$ and base $\beta_1 - \sigma(\beta_1)$. Since F_{Y_1} is weakly increasing, if $\sigma(\beta_1)$ were to be equal to $\underline{v}$, then the area under the curve would lie entirely inside the rectangle so the equality fails. Thus, the value of $\sigma(\beta_1)$ must be sufficiently above $\underline{v}$ to bring about an equality.

Contrast this to the ETE of the English auction. Substituting (3) into (8) yields,

$$\lim_{\delta \to 1} \int_{\sigma(\beta_1)}^{\beta_1} F_{Y_1}(Y_1) dY_1 = \int_{\underline{v}}^{\beta_1} F_{Y_1}(Y_1) \, dY_1.$$

For this equality to hold, $\sigma(\beta_1)$ must be equal to $\underline{v}$ in the limit. The distinction is that unlike in the second-price auction, the seller cannot screen out buyers whose valuations exceed the starting price since interim bidders are free to enter the bidding after the opening bid. Thus the marginal bidder pays $Y_1 > \sigma(\beta_1)$ for realizations of Y_1 in $[\sigma(\beta_1), \beta_1)$. Under those circumstances, the surplus received by the marginal bidder is equal to that of a one-shot auction with starting price $\sigma(\beta_1)$. A buyer comparing the surplus from a one-shot auction with starting price $\sigma(\beta_1)$ to that of a one-shot auction with starting price $\underline{v}$ will choose the former only if $\sigma(\beta_1) = \underline{v}$. Thus the only way the seller can guarantee the type-β_1 buyer his reservation surplus is by running an auction with a starting price close to $\underline{v}$.

Another important difference between the two auction formats is that the second-price auction does not have the strong gap property. The strong gap property in the sequential English auction follows from the fact that the initial starting price converges to zero as the time between auctions becomes short. Conditional upon reaching the terminal period in the English auction, the second-highest buyer's valuation exceeds the initial starting price with probability approaching one. In the second-price auction, the initial starting price is bounded above $\underline{v}$ and as a result may exceed the second-highest buyer's valuation.

6 Conclusion

This paper began with the task of understanding the bidding dynamics that result when an auctioneer fishes for an opening bid. We have studied a model of an English auction in which the seller goes fishing for an opening bid when she cannot commit to a predetermined starting-price path. The outcome observed by Cassady (1967), in which the price in a subsequent auction exceeds the amount of the starting price in the initial auction, is shown to be a natural consequence of the model. Further, it is shown to be the most likely outcome when the time between successive auctions is sufficiently short.

An implication of the model is that when auctions are conducted sequentially, the English auction is no longer strategically equivalent to the second-price auction. The English auction is shown to induce bids from a larger set of buyer types than does the second-price auction. The difference is in the participation of *interim bidders*, who enter the bidding only after first observing a bid from a competing buyer. Because of the incentive for buyers to take a wait-and-see approach, the seller in the sequential English auction must lower the sequence of starting prices from what they would be were the auction mechanism a second-price auction. The sequence of starting prices is set in such a way as to equalize revenues across the two formats as well as to make the probability of sale and the allocation of the item identical. Since the participation of interim bidders in the English auction allows the seller to lower her starting price while keeping revenues unchanged, one could say that the interim bidders are "doing the seller's bidding."

A key finding of the paper is that the seller's initial starting price converges to the lowest buyer valuation when the time between auctions is sufficiently short. This suggests that the act of fishing has very little value to the seller. The result is analogous to the Coase conjecture in the sequential bargaining or durable goods monopoly settings. It is interesting to note that while the inability to commit to a starting/reserve price policy has the same impact on revenue when the seller conducts second-price or first-price auctions, it does not force the seller's initial reserve price to its lowest possible level. As a consequence, the *start low end high* phenomenon reported by Cassady could not be explained by treating the English auction as equivalent to the second-price auction.

Of course the seller's revenues would be greater could she commit to not relist the item if the initial auction fails to induce any bids. It is then natural to ask why institutions have not arisen to solve the seller's commitment problem. After all, McAdams and Schwartz (2007) argue that a role for the auction house is to eliminate commitment problems that prevent mutually beneficial transactions from taking place. But in the current setting, a seller's inability to commit to a starting-price path is efficiency enhancing as it forces the seller to offer the item at a low initial starting price, thereby insuring the item ends up in the hands of whomever values it the most. The auction house would have an interest in the efficient outcome since in many instances, the auction house is responsible for assembling the group of bidders. It should therefore not be surprising that opening bid fishing persists.

An interesting empirical question is whether dynamic considerations are responsible for the *start low end high* phenomenon described by Cassady and not other explanations such as affiliated values or auction fever. The model provides a testable prediction between the time to relisting and the probability of having a subsequent auction end in a price of at least S after the initial auction started at S fails to sell: the shorter is the time to relisting, the more likely is a subsequent auction to reach a price of S. In contrast, considerations of affiliated values or auction fever would reasonably be unaffected by the time to relisting. The implementation of such a test is left to future research.

APPENDIX

A Proofs

A.1 Proof of Lemma 1

1. Fix a starting price s_t and for bidder 1, say, let dB_1 denote the density of the highest of maximum bid prices of all other $n-1$ buyers should buyer 1 submit a bid. Upon bidding at the starting price, the auction will necessarily result in a sale. The expected return to bidder 1 of playing a strategy of bidding up to some amount b is

$$(v - s_t) \int_0^{s_t} dB_1 + \int_{s_t}^{b} (v - B_1)\, dB_1,$$

where we let v denote buyer 1's valuation. This expression is maximized at $b = v$ for any set of strategies giving rise to the arbitrary density dB_1.

2. The proof proceeds to show that if some type $v > s_t$ is an initial bidder, then so too is any type $v' > v$. Assuming that buyer 1, upon bidding at the starting price, bids up to v and let dB_1 denote the highest of maximum bid prices of all other $n-1$ bidders in the current period. Let $V_B(z; v, H_t)$ denote the continuation payoff of a type v buyer from the following period on, given history H_t, playing the strategy of a type-z buyer in what follows. Further, let dA_1 denote the density of the highest of maximum bid prices of all other buyers should buyer 1 abstain from bidding. If a type-v buyer is an initial bidder in period t, then

$$(v - s_t) \int_0^{s_t} dB_1 + \int_{s_t}^{v} (v - B_1)\, dB_1 \geq \delta V_B(v; v, H_t) \int_0^{s_t} dA_1. \tag{9}$$

Expression (9) states that the surplus from bidding up to the buyer's valuation must exceed the expected value of not bidding.

Now suppose, by way of contradiction, that some type $v' > v$ finds it unprofitable to bid at the start price in period t. This implies that

$$(v' - s_t) \int_0^{s_t} dB_1 + \int_{s_t}^{v} (v' - B_1)\, dB_1 < \delta V_B(v'; v', H_t) \int_0^{s_t} dA_1. \tag{10}$$

Since a type v buyer can always adopt the strategy of type v', incentive compatibility implies

$$\begin{aligned} V_B(v; v, H_t) &\geq V_B(v'; v, H_t) \\ &= \sum_{j=0}^{\infty} \delta^j \alpha_{t+1+j}(v') \left[v - m_{t+1+j}(v') \right], \end{aligned}$$

where $\alpha_{t+1+j}(v'; H_t)$ denotes the probability, conditional on H_{t+j}, that the item is obtained in period $t+1+j$ playing the strategy of type v' and $m_{t+1+j}(v'; H_t)$ is the analogous expected payment. It follows that

$$V_B(v'; v', H_t) - V_B(v; v, H_t) \leq (v' - v) \sum_{j=0}^{\infty} \delta^j \alpha_{t+1+j}(v'; H_t). \tag{11}$$

From (9) and (10), we have that

$$\begin{aligned}(v' - v) \int_0^v dB_1 &< \delta [V_B(v'; v', H_t) - V_B(v; v, H_t)] \int_0^{s_t} dA_1 \\ &< (v' - v) \delta \sum_{j=0}^{\infty} \delta^j \alpha_{t+1+j}(v'; H_t) \int_0^{s_t} dA_1, \end{aligned} \tag{12}$$

where the second inequality follows from (11). Equation (12) necessarily leads to a contradiction as long as $\int_0^v dB_1 \geq \int_0^{s_t} dA_1$ since $\sum_{j=0}^{\infty} \delta^j \alpha_{t+1+j}(v'; H_t)$ can be no greater than 1.

Now, $\int_0^v dB_1$is the probability of obtaining the item for the type$-v$ buyer and $\int_0^{s_t} dA_1$ is the probability that the item goes unsold when the buyer in question abstains from bidding. So too, $\int_0^{s_t} dB_1$ is the probability of obtaining the item for a type$-s_t$ buyer. Since a type s_t buyer wins only when he is the lone bidder, we have that

$$\int_0^{s_t} dB_1 = \int_0^{s_t} dA_1. \tag{13}$$

It follows from (13), that if we choose v to be some increment greater than s_t and increase the upper integrand on the left-hand side of (13) by that increment, we have that $\int_0^v dB_1 \geq \int_0^{s_t} dA_1$.

Since v' was chosen arbitrarily, it must be the case that if some type v submits a bid in period t, then so does every buyer whose valuation exceeds v.

3. We begin by asserting that there exists a minimum starting price such that all bidder types bid whenever the starting price is less than or equal to the minimum, regardless of the history. The claim is that $\underline{v}$-$\bar{v}$ is one such starting price. We know that in equilibrium, the seller's expected receipts must be nonnegative–since she can always opt not to sell–and that a buyer's expected surplus cannot exceed $\bar{v}$ by the same token. Therefore, the expected surplus for a buyer with valuation $\underline{v}$ is *at most* $\underline{v}$ minus the starting price. This is less than $\bar{v}$ as long as the starting price is less than $\underline{v}$-$\bar{v}$. Thus, all types bid when the starting price is less than or equal to $\underline{v}$-$\bar{v}$.

We now calculate a buyer's expected surplus at the minimum starting price. When all buyer types bid and the starting price is less than $\underline{v}$, a given buyer's expected surplus is $\int_{\underline{v}}^{v} F_{Y_1}(y)\, dy \geq 0$. Notice that a buyer's expected surplus is independent of the actual starting price as the price will necessarily be determined by the bid of the second-highest valuation buyer. This is crucial in what follows.

We work recursively to show that the minimum starting price is in fact $\underline{v}$. Consider a starting price, $s_{\varepsilon_1} = \underline{v} - \bar{v} + \varepsilon$, just slightly greater than $\underline{v}$-$\bar{v}$, such that if the auction started at s_{ε_1} fails to sell, the starting price is reduced to $\underline{v}$-$\bar{v}$ in the following period. When the starting price is s_{ε_1}, a given buyer bids at the start as long as the surplus gained in the current period exceeds the surplus gained in the following period should the item go unsold. Assume by way of contradiction that there exists some $\beta_{\varepsilon_1} > \underline{v}$ that is the lowest type to bid at the starting price. Consider then a buyer with valuation $v < \beta_{\varepsilon_1}$. That the valuation$-v$ buyer does not bid in the current period, it must be that,

$$[v - \rho(s_{\varepsilon_1}, v)] F_{Y_1}(\beta_{\varepsilon_1}) < \delta \int_{\underline{v}}^{v} F_{Y_1}(Y_1)\, dY_1. \tag{14}$$

The left-hand side of (14) indicates the type$-v$ buyer's surplus in the current period, where $\rho(s_{\varepsilon_1}, v)$ denotes his expected payment, taking into account that interim bidders' highest bid price. Interim bidders' strategies are as yet undetermined, but we assume that they do not play any dominated strategies such as bidding above their valuation. The right-hand side of (14) indicates the type$-v$ buyer's surplus in the following period, where the starting price is reduced to $\underline{v} - \bar{v}$ and all remaining buyer types bid.

As the mixed strategy followed by interim bidders is yet undetermined, so is $\rho(s_{\varepsilon_1}, v)$. However, the highest payment made by a type-v buyer is when all interim bidders bid up to their valuations, which implies,

$$\rho(s_{\varepsilon_1}, v) = E[Y_1 | Y_1 < v].$$

Combining this with the left-hand side of (14), we have that

$$[v - \rho(s_{\varepsilon_1}, v)] F_{Y_1}(\beta_{\varepsilon_1}) > \int_{\underline{v}}^{v} F_{Y_1}(Y_1)\, dY_1. \tag{15}$$

Combining (14) and (15), we have that,

$$\int_{\underline{v}}^{v} F_{Y_1}(Y_1)\, dY_1 < \delta \int_{\underline{v}}^{v} F_{Y_1}(Y_1)\, dY_1,$$

which is a contradiction. Therefore, it cannot be the case that $\beta_{\varepsilon_1} > \underline{v}$.We conclude that all types bid when the starting price is s_ε or less.

For the inductive step, consider a starting price $s_{\varepsilon_k} = s_{\varepsilon_{k-1}} + \varepsilon < \underline{v}$, such that if the item fails to sell at s_{ε_k} the seller reduces the starting price to $s_{\varepsilon_{k-1}}$in the following period, wherein all remaining buyers will bid. As before, assume by way of contradiction that there exists some $\beta_{\varepsilon_k} > \underline{v}$ that is the lowest type to bid. Consider then a buyer with valuation $v < \beta_{\varepsilon_k}$. That the valuation$-v$ buyer does not bid in the current period, it must be that,

$$[v - \rho(s_{\varepsilon_k}, v)] F_{Y_1}(\beta_{\varepsilon_1}) < \delta \int_{\underline{v}}^{v} F_{Y_1}(Y_1)\, dY_1. \tag{16}$$

As before, the left-hand side can be minimized by substituting

$$\rho(s_{\varepsilon_k}, v) = E[Y_1 | Y_1 < v]$$

into (16). This leads to condition (15) only with β_{ε_k} in place of β_{ε_1}, which contradicts the assumption that $\beta_{\varepsilon_k} \geq \underline{v}$. We then conclude that all types bid when the starting price is s_{ε_k} or less. This establishes the result. Note that the recursion does not extend to $s_{\varepsilon_{k+1}} > \underline{v}$ since such starting prices may actually determine the price, meaning that a buyer's participation decision does not give rise to equation (15).

4. Let $g\left(u_t, \beta_t, \beta_{t+1}\right)$ denote the seller's expected payoff in state u_t, when choosing a starting price in the current period that induces a screening level of β_t, which subsequently induces a screening level of β_{t+1} in the following period. For ease of notation let Y_1 denote the highest valuation among all buyers other than buyer 1 (the reference buyer) and let $F_{Y_1} \equiv F^{n-1}$ denote the distribution of Y_1.Note that from part 2 of the lemma, β_t exceeds β_{t+1} and from part 3, β_τ is equal to $\underline{v}$ if $s_\tau \leq \underline{v}$. We have that,[22]

$$\begin{aligned} g\left(u_t, \beta_t, \beta_{t+1}\right) &= n\rho_t\left(s_t, \beta_t\right)\left[F\left(u_t\right) - F\left(\beta_t\right)\right] F_{Y_1}\left(\beta_t\right) \\ &\quad + n \int_{\beta_t}^{u_t} \int_{\beta_t}^{X_1} Y_1 dF_{Y_1} f\left(X_1\right) dX_1 \\ &\quad + \delta \Gamma_{t+1}\left(\beta_t\right), \end{aligned}$$

where $\Gamma_{t+1}\left(\beta_t\right)$ is the seller's optimal payoff from period $t+1$ on, beginning at state β_t. The first-order condition for the seller's optimal choice of β_t reduces to

$$(1-\delta)\left[F\left(u_t\right) - F\left(\beta_t\right) - \beta_t f\left(\beta_t\right)\right] F_{Y_1}\left(\beta_t\right) \leq 0.$$

Since $f(\cdot)$ is positive, there exists some $u^* > \underline{v}$ such that for $u_t < u^*$, $F\left(u_t\right) - F\left(\beta_t\right) - \beta_t f\left(\beta_t\right)$ is strictly negative. Thus for $u < u^*$, it is optimal for the seller to induce bids from all types thus ending the game.

Next we show that the seller's beliefs fall below u^* in finite time. For this, we again examine the seller's first-order condition for the optimal screening level. Solving for an interior optimum and rearranging terms yields

$$F\left(u_t\right) - F\left(\beta_t\right) = \beta_t f\left(\beta_t\right).$$

Since f is bounded away from zero, so too is the distance between u_t and β_t. So in an interior optimum, implying $u_t > u^*$, the screening level jumps down in discrete steps so that some $u^* > \underline{v}$ is eventually reached. If the optimum is not an interior solution, then by definition, $u_t < u^*$.

A.2 Proof of Proposition 1

We begin by defining a $k-$period game in which the seller and each buyer behaves optimally given the constraint that should the item fail to sell, the game necessarily ends after $k-1$ periods. For this constrained game, denote the screening level, seller's starting price, and expected revenue in the terminal period:

$$\beta_0 \equiv \underline{v},\ s_0 \equiv \underline{v},\ \Gamma_0(u) \equiv n \int_{\underline{v}}^{u} \int_{\underline{v}}^{X_1} Y_1 dF_{Y_1} f\left(X_1\right) dX_1.$$

[22] For a more thorough explanation, see the discussion following expression (18).

Expected revenue is calculated by considering first the expected payment of some buyer, say buyer 1, with valuation X_1. Since the auction in the final period has a starting price of $\underline{v}$, buyer 1's price will be determined by the maximum of $n-1$ valuations of other buyers, denoted Y_1. Buyer 1's expected payment is the expectation of Y_1 over $[\underline{v}, X_1]$, where $F_{Y_1} \equiv F^{n-1}$ denotes the distribution of Y_1. The seller's expected revenue is simply n-times the expectation of a given buyer's expected payment.

Define the sequences

$$\left\{\beta_j\right\}_{j=0}^k, \ \left\{\sigma_j\right\}_{j=0}^k, \ \left\{\rho_j\right\}_{j=0}^k, \ \left\{\Gamma_j\right\}_{j=0}^k, \ \left\{g_j\right\}_{j=0}^k$$

iteratively as follows.

Let $\sigma_j(x)$ denote the starting price that induces a screening level of x in the jth-to-last period of the k period game. Since the type-x buyer wins the auction only upon being the lone initial bidder, $\sigma_j(x)$ and $\rho_j(\sigma_j(x), x)$ satisfy

$$\left[x - \rho_j(\sigma_j(x), x)\right] F_{Y_1}(x) = \delta\left(\left[\beta_{j-1} - \rho_{j-1}\right] F_{Y_1}\left(\beta_{j-1}\right) + \int_{\beta_{j-1}}^{x} F_{Y_1}(Y_1)\, dY_1\right). \tag{17}$$

The left-hand side of the expression indicates a type-x buyer's expected surplus from bidding at starting price $\sigma_j(x)$, when he is the lowest type to be an initial bidder. His expected payment under the circumstance is $\rho_j(\sigma_j(x), x)$. The right-hand side of the expression gives a type-x buyer's expected continuation surplus, given a starting price of σ_{j-1} and a screening level of β_{j-1} in the period to follow. In the following period, since $\beta_j \geq \beta_{j-1}$, he receives an amount given by the first term, where $\rho_{j-1} \equiv \rho_j\left(\sigma_{j-1}, \beta_{j-1}\right)$, in the event that he is the lone initial buyer and an amount given by the second term when he is bidding against at least one other initial bidder.

Let $g_j(u, x)$ denote the seller's revenue in the jth-to-last period, at state u, when choosing a starting price that induces a screening level of x. We have that

$$\begin{aligned} g_j(u, x) &= n\rho_j(\sigma_j(x), x) F_{Y_1}(x)\left[F(u) - F(x)\right] + n\int_x^u \int_x^{X_1} Y_1 dF_{Y_1} f(X_1)\, dX_1 \\ &\quad + \delta\Gamma_{j-1}(x). \end{aligned} \tag{18}$$

The first term represents the seller's revenue from having only one initial buyer and the second from having at least two. The third term represents her maximum discounted return from the following period on, at state x, in the event the current auction fails to produce a sale. The seller solves the problem,

$$\Gamma_j(u) = \max_{x \leq u} g_j(u, x), \tag{19}$$

of which β_j is the solution, satisfying

$$\beta_j = \arg\max_{x \leq u} \left\{g_j(u, x)\right\}. \tag{20}$$

Lemma 2 *For a given $k > 1$, the sequences $\left\{\beta_j\right\}_{j=0}^{k}$, $\left\{\rho_j\right\}_{j=0}^{k}$, $\left\{\Gamma_j\right\}_{j=0}^{k}$ are such that:*

1. *The $\sigma_j < x$ satisfying* (17) *are unique and increasing in x for $x > \underline{v}$.*
2. *The $\Gamma_j(x)$ satisfying* (19) *are increasing and continuous.*
3. *The $\beta_j(u)$ satisfying* (20) *are strictly less than u and increasing.*

Proof. Property 1 is proven directly from (17). Uniqueness follows from the fact that the left-hand side of (17) is strictly decreasing in σ while the right-hand side is constant in σ for $x > \underline{v}$ thus implying a single point of intersection. Differentiating both sides of (17) with respect to x and rearranging, we have

$$\frac{d\sigma_j}{dx} = \frac{(1-\delta) F_{Y_1}(x) + \left[x - \rho_j(\sigma_j, x)\right] f_{Y_1}(x)}{\frac{\partial \rho_j(\sigma_j, x)}{\partial \sigma_j} F_{Y_1}(x)}.$$

The numerator of this expression is positive by the fact that $\rho_j \leq x$. The denominator is positive under the assumption that $\rho_j(\sigma_j, x)$ is increasing in σ_j.

Properties 2 and 3 are proven by induction. It is straightforward to show that both of properties 2 and 3 are satisfied for $j = 2$. Now assume, by way of induction, that each of properties 2 and 3 are satisfied for $j = k-1$. Since $\rho_j(\sigma_k(x), x)$ is continuous in both arguments and $\sigma_k(x)$ is continuous in x, then $g_k(u, x)$ is continuous in both arguments. It follows using standard arguments that Γ_k is continuous and increasing.

Consider now $u < u'$ and let $x \in \arg\max_y \{g_k(u, y)\}$and let $x' \in \arg\max_y \{g_k(u', y)\}$. Suppose, by way of contradiction, that $x' < x$. We have that

$$\begin{aligned} g_k(u', x) &= g_k(u, x) + n\rho_k(\sigma_k(x), x) F_{Y_1}(x)\left[F(u') - F(u)\right] \\ &\quad + n\int_u^{u'}\int_x^{X_1} Y_1 dF_{Y_1}(Y_1) f(X_1) dX_1 \end{aligned} \tag{21}$$

and

$$\begin{aligned} g_k(u', x') &= g_k(u, x') + n\rho_k(\sigma_k(x'), x') F_{Y_1}(x)\left[F(u') - F(u)\right] \\ &\quad + n\int_u^{u'}\int_{x'}^{X_1} Y_1 dF_{Y_1}(Y_1) f(X_1) dX_1. \end{aligned} \tag{22}$$

Subtracting (22) from (21), we have

$$\begin{gathered} g_k(u', x) - g_k(u', x') - \left[g_k(u, x) - g_k(u, x')\right] = \\ n\left[\rho_k(\sigma_k(x), x) F_{Y_1}(x) - \rho_k(\sigma_k(x'), x') F_{Y_1}(x') - \int_{x'}^{x} Y_1 dF_{Y_1}(Y_1)\right]\left[F(u') - F(u)\right]. \end{gathered} \tag{23}$$

The left-hand side of (23) is non-positive since x is a maximizer of $g_k(u, \cdot)$ and x' is a maximizer of $g_k(u', \cdot)$. We now want to show the right-hand side of (23) to be positive, so resulting in a contradiction.

For compactness, let $\Upsilon \equiv \rho_k(\sigma_k(x), x) F_{Y_1}(x)$ and $\Upsilon' \equiv \rho_k(\sigma_k(x'), x') F_{Y_1}(x')$. Using this notation, the right-hand side of (23) is positive if

$$\Upsilon - \Upsilon' - \int_{x'}^{x} Y_1 dF_{Y_1}(Y_1) \geq 0. \tag{24}$$

From the period$-k$ analogue of (18), we have

$$\Upsilon = (1-\delta) x F_{Y_1}(x) + \delta \Upsilon_{k-1} + \delta \int_{\beta_{k-1}}^{x} Y_1 dF_{Y_1}(Y_1),$$

where $\Upsilon_{k-1} \equiv \rho_{k-1}\left(\sigma_{k-1}\left(\beta_{k-1}\right), \beta_{k-1}\right) F_{Y_1}\left(\beta_{k-1}\right)$ and $\beta_{k-1} \equiv \beta_{k-1}(x)$. Using the fact that, from (18), $d\Upsilon / d\beta_{k-1} \geq 0$,

$$\begin{aligned}
\Upsilon &\geq (1-\delta) x F_{Y_1}(x) + \delta \Upsilon'_{k-1} + \delta \int_{\beta'_{k-1}}^{x} Y_1 dF_{Y_1}(Y_1) \\
&= \Upsilon' + (1-\delta)\left[x F_{Y_1}(x) - x' F_{Y_1}(x')\right] + \delta \int_{x'}^{x} Y_1 dF_{Y_1},
\end{aligned}$$

where $\beta'_{k-1} < \beta_{k-1}$ and so too $\Upsilon'_{k-1} < \Upsilon_{k-1}$ by the assumption that $x' < x$. It follows that (24) holds if

$$(1-\delta)\left(\left[x F_{Y_1}(x) - x' F_{Y_1}(x')\right] - \int_{x'}^{x} Y_1 dF_{Y_1}\right) \geq 0.$$

The above is true under the assumption that $x' < x$, thus yielding the desired contradiction. ■

Having characterized the equilibrium to the arbitrarily constrained k period game, we can extend the results of Lemma 2 to the unconstrained game. We show that at any stage of the game, the number of remaining periods is determined solely by u, the highest potential buyer valuation. This is done by constructing a sequence of numbers $\{z_j\}_{j=0}^{T}$ iteratively as follows. Let

$$z_1 = \sup\left\{u | \beta_1(u) = \underline{v}\right\}$$

denote the largest value of u such that the seller chooses to end the game immediately and

$$z_j = \min\left\{\sup\left\{u | \beta_j(u) \leq z_{j-1}\right\}, \bar{v}\right\}$$

denote the largest value of u such that the seller chooses a screening level in the current period such that the optimal policy from the following period onward has her end the game in $j-1$ periods. The following lemma shows that there exists some T such that when $u = \bar{v}$, the seller chooses a screening level such that the optimal policy from the following period onward has her end the game in $T-1$ periods.

Lemma 3 *There exists an $\varepsilon > 0$ such that for all ρ, δ, and n, $z_1 \geq \underline{v} + \varepsilon$. Further, there exists a $T < \infty$ such that $z_T = \bar{v}$.*

The proof is identical to that of Lemma 2 in McAfee and Vincent (1997), only with ρ taking the place of σ, so there is no need to repeat it here.

With the z_j so defined, we can define the seller's problem uniquely by u, independent of j. In this way, if $u \in (z_{j-1}, z_j]$, the seller chooses the optimal screening level independent of j; it just so happens that such a screening level will lead, assuming optimal behavior in what follows, to the game ending in $j-1$ more periods should the item fail to sell. In what follows, we change our notational convention so that a subscript t denotes $(t-1)$ periods *after* the initial period as opposed to t periods *before* the terminal period. In this way, given $u_1 = \bar{v}$, we have that $u_2 = \beta(\bar{v})$, $u_t = \beta(u_{t-1})$ and $s_t = \sigma(\beta_t)$ for any $t > 1$.

The following addresses the three individual components of Proposition 1.

Taking as given the sequence of screening levels, $\{\rho_t\}$ is the sequence of the expected payment made by the marginal bidder type in each period. From (17), ρ_{T-1} satisfies

$$\left(\beta_{T-1} - \rho_{T-1}\right) F_{Y_1}\left(\beta_{T-1}\right) = \delta \int_{\underline{v}}^{\beta_{T-1}} F_{Y_1} dY_1.$$

Working backward, we that in any period $t < T-1$, ρ_t satisfies

$$\left(\beta_t - \rho_t\right) F_{Y_1}\left(\beta_t\right) = \delta \left(\left[\beta_{t+1} - \rho_{t+1}\right] F_{Y_1}\left(\beta_{t+1}\right) + \int_{\beta_{t+1}}^{\beta_t} F_{Y_1}\left(Y_1\right) dY_1 \right), \qquad (25)$$

so that the sequence $\{\rho_t\}$ is unique to a given sequence of $\{\beta_t\}$. The payment by the marginal type is a known function, $\rho_t(\cdot,\cdot)$ of the starting price and screening level. Therefore, for the optimal sequence of screening levels $\{\beta_t\}$ and the corresponding sequence of payments $\{\rho_t\}$, an equilibrium starting price $\sigma(\beta_t)$ is some price such that $\rho_t(\sigma(\beta_t), \beta_t) = \rho_t$. Under the assumption that ρ_t be increasing in both arguments, $\sigma(\beta_t)$ is unique and increasing in β_t. This establishes part 2 of the proposition.

The seller's revenue in period T is

$$g\left(u_T, \underline{v}\right) = n \int_{\underline{v}}^{u_T} \int_{\underline{v}}^{X_1} Y_1 dF_{Y_1} f\left(X_1\right) dX_1.$$

Working backward, the solution to the sellers problem gives rise to the first-order condition

$$\beta_t f\left(\beta_t\right) + F\left(\beta_t\right) - F\left(u_t\right) = 0$$

which characterizes a solution in u, independent of n, δ, and ρ. This establishes part 1 of the proposition.

Since the seller continues to relist until a sale is transacted, the item is allocated to the buyer with the highest valuation, regardless of ρ_t. The probability of sale in some period t is $F\left(\beta_{t-1}\right)^n - F\left(\beta_t\right)^n$. Since the sequence of screening levels is the same for each value of ρ_t, so is the probability of sale in each period. The seller's revenue in a given period is given by the first two terms in $g\left(u_t, \beta_t\right)$. Using the fact that $u_t = \beta_{t-1}$, after some manipulation, we have the seller's period-t revenue is

$$\begin{aligned} R\left(\beta_{t-1}, \beta_t\right) &= n \int_{\beta_t}^{\beta_{t-1}} \left[v f(v) + F(v) - F\left(\beta_{t-1}\right)\right] F_{Y_1}(v)\, dv \\ &\quad - n \left[F\left(\beta_{t-1}\right) - F\left(\beta_t\right)\right] \left(\beta_t - \rho_t\right) F_{Y_1}\left(\beta_t\right). \end{aligned}$$

Substituting in from (25) recursively $(T-t)$ times, we have

$$\begin{aligned} R\left(\beta_{t-1}, \beta_t\right) &= n \int_{\beta_t}^{\beta_{t-1}} \left[v f(v) + F(v) - F\left(\beta_{t-1}\right)\right] F_{Y_1}(v)\, dv \\ &\quad - n \left[F\left(\beta_{t-1}\right) - F\left(\beta_t\right)\right] \sum_{j=0}^{T-t} \delta^{j+1} \int_{\beta_{t+j+1}}^{\beta_{t+j}} F_{Y_1}(Y_1)\, dY_1, \end{aligned}$$

which is independent of ρ_t.

A.3 Proof of Proposition 2

The effect of the trembles on an interim bidders is to create the possibility of winning the auction when bidding no more than his valuation. This is because 1. the buyer who bid at the starting price may simply be an interim bidder who bid by mistake; or 2. all true initial bidders may mistakenly drop out of the bidding before their valuations are reached. It is sufficient to show that given the possibility of obtaining the item, all interim bidders' strategies other than the one proposed are weakly dominated.

Consider bidder 1, an interim bidder with valuation $v \geq s_t$, and let dB_ε denote the density of the highest maximum bid price of all $n-1$ other buyers, given that a bid was placed at the starting price. Note that if the probability of all trembles were zero to be zero, dB_ε would be equal to dF_{Y_1}. Suppose buyer 1 chooses some maximum price b at which to drop out. His expected payoff from that strategy is

$$\int_{s_t}^{b} \left(v - B_\varepsilon\right) dB_\varepsilon.$$

This expression is maximized at $b = v$ for any profile of behavioral strategies played by the other bidders which gives rise to dB_ε.

A.4 Proof of Proposition 3

Suppose the seller wishes to induce a screening level of β_1 in the initial period. If $\beta_1 = \underline{v}$, then she simply chooses a starting price no greater than $\underline{v}$ and the result is shown. Assume then that $\beta_1 > \underline{v}$. To induce β_1, the seller chooses a reserve $\sigma\left(\beta_1\right)$ giving rise to $\rho_1 \equiv \rho\left(\sigma\left(\beta_1\right), \beta_1\right)$, solving

$$\left(\beta_1 - \rho_1\right) F_{Y_1}\left(\beta_t\right) = \delta \left[\int_{\beta_2}^{\beta_1} F_{Y_1} dY_1 + \left(\beta_2 - \rho_2\right) F_{Y_1}\left(\beta_2\right)\right], \qquad (26)$$

where $\beta_2 = \beta(\beta_1)$ and $\rho_2 = \rho(\sigma(\beta_2), \beta_2)$. By the same logic, the second term on the right-hand side of (26), assuming $\beta_2 > \underline{v}$ satisfies

$$(\beta_2 - \rho_2) F_{Y_1}(\beta_2) = \delta \left[\int_{\beta_3}^{\beta_2} F_{Y_1} dY_1 + (\beta_3 - \rho_3) F_{Y_1}(\beta_3) \right].$$

Following this logic recursively, and noting that

$$\left(\beta_{T-1} - \rho_{T-1}\right) F_{Y_1}\left(\beta_{T-1}\right) = \delta \int_{\underline{v}}^{\beta_{T-1}} F_{Y_1} dY_1,$$

since $\beta_T = \underline{v}$, (26) becomes

$$(\beta_1 - \rho_1) F_{Y_1}(\beta_t) = \delta \sum_{j=0}^{T-(t+1)} \delta^j \int_{\beta_{t+j+1}}^{\beta_{t+j}} F_{Y_1} dY_1. \tag{27}$$

Using (3) on the left-hand side of (27),

$$(\beta_1 - \rho_1) F_{Y_1}(\beta_t) = \int_{\sigma(\beta_1)}^{\beta_1} F_{Y_1} dY_1. \tag{28}$$

We are interested in the value of $\sigma(\beta_1)$ as δ gets arbitrarily close to unity. Therefore, in (27),

$$\lim_{\delta \to 1} \delta \sum_{j=0}^{T-(t+1)} \delta^j \int_{\beta_{t+j+1}}^{\beta_{t+j}} F_{Y_1} dY_1 = \int_{\underline{v}}^{\beta_1} F_{Y_1} dY_1. \tag{29}$$

Putting (28) together with (29), (27) implies

$$\lim_{\delta \to 1} \int_{\sigma(\beta_1)}^{\beta_1} F_{Y_1} dY_1 = \int_{\underline{v}}^{\beta_1} F_{Y_1} dY_1.$$

The only way this can hold is if $\sigma(\beta_1) \to \underline{v}$.

A.5 Proof of Proposition 4

Consider the auction in period $t+1$ assuming all prior auctions failed to induce any bids. From equation (2), we have that $\beta_t > \rho_t > s_t$. Therefore, the fact that all auctions conducted through period t failed to induce any bids implies that the highest possible valuation as of period $t+1$ is β_t. It follows that

$$\begin{aligned} \Pr\{p_{t+1} > s_t | \eta_t = 0\} &= \Pr\left\{X_1 \geq \beta_{t+1}, X_2 > s_t | X_1 < \beta_t\right\} \\ &> \Pr\{X_1 \geq \beta_t, X_2 > s_t | X_1 < \beta_t\} \\ &= 0 \end{aligned}$$

where the inequality in the second line follows from the fact that $\beta_t > \beta_{t+1}$ (which was established in Lemma 1. This establishes that $\bar{\tau}$ is at least unity.

A.6 Proof of Proposition 5

Consider the equilibrium of a game with an arbitrary number of periods, denoted T. Conditional upon period T being reached, the highest possible valuation is β_{T-1}. In period T, the starting price is reduced to $\underline{v}$, so that all buyers bid up to their valuations. It follows that the probability that the period$-T$ auction ends with a price exceeding the period-t starting price, $s_t \equiv \sigma(\beta_t)$, is the probability that the second-highest buyer valuation is at least s_t. Formally,

$$\Pr\left\{p_T > s_t | \eta_{T-1} = 0\right\} = \Pr\left\{X_2 > s_t | X_1 < \beta_{T-1}\right\}. \tag{30}$$

In the limit as $\delta \to 1$, we have from Proposition 3 that $\sigma(\beta_t) \to \underline{v}$. Combining this with (30), we have that,

$$\begin{aligned}\lim_{\delta\to 1} \Pr\left\{p_T > s_t | \eta_{T-1} = 0\right\} &= \Pr\left\{X_2 > \underline{v} | X_1 < \beta_{T-1}\right\} \\ &= 1.\end{aligned}$$

In period t, buyers realize that if no bids are received in that period's auction, they may obtain the item at a lower price in a subsequent auction. Therefore, a bid is placed at the starting price only if there is at least one bidder whose valuation exceeds β_t. Therefore, the probability that the price exceeds s_t is,

$$\Pr\left\{p_t > s_t | \eta_{t-1} = 0\right\} = \Pr\left\{X_1 \geq \beta_t, X_2 > s_t | X_1 < \beta_{t-1}\right\}. \tag{31}$$

As $\delta \to 1$, this becomes,

$$\begin{aligned}\lim_{\delta\to 1} \Pr\left\{p_t > s_t | \eta_{t-1} = 0\right\} &= \Pr\left\{X_1 \geq \beta_t, X_2 > \underline{v} | X_1 < \beta_{t-1}\right\} \\ &= \Pr\left\{X_1 \geq \beta_t | X_1 < \beta_{t-1}\right\} \\ &< 1\end{aligned}$$

where the inequality in the third line follows from Result 1 of Proposition 1 which says that the sequence $\{\beta_t\}$ is independent of δ, and that $\beta_t < \beta_{t-1}$. It follows that for δ sufficiently close to 1, the quantity in (30) exceeds the quantity in (31).

REFERENCES

Ausubel, L. and R. Deneckere (1989). Reputation in bargaining and durable goods monopoly. *Econometrica 57*, 511–531.

Bester, H. and R. Strausz (2001). Contracting with imperfect commitment and the revelation principle. *Econometrica 69*, 1077–1098.

Burguet, R. and J. Sakovics (1996). Reserve prices without commitment. *Games and Economic Behavior 15*, 149–164.

Caillaud, B. and C. Mezzetti (2004). Equilibrium reserve prices in sequential ascending auctions. *Journal of Economic Theory 117*, 78–95.

Cassady, R. (1967). *Auctions and Auctioneering*. Berkeley, CA: University of California Press.

Coase, R. (1972). Durability and monopoly. *Journal of Law and Economics 15*, 143–149.

Gul, F., H. Sonnenschein, and R. Wilson (1986). Dynamic monopoly. *Journal of Economic Theory 39*, 155–190.

Ku, G. ., A. Galinsky, and J. Murninghan (2006). Starting low but ending high: A reversal of the anchoring effect in auctions. *Journal of Personality and Social Psychology 90*, 975–986.

McAdams, D. and M. Schwartz (2007). Credible sales mechanisms and intermediaries. *American Economic Review 97*, 260–276.

McAfee, R. and D. Vincent (1997). Sequentially optimal auctions. *Games and Economic Behavior 18*, 246–276.

Meyerson, R. (1981). Optimal auction design. *Mathematics of Operations Research 6*, 58–73.

Milgrom, P. and R. Weber (1982). A theory of auctions and competitive bidding. *Econometrica 50*, 1089–1122.

Ockenfels, A., D. Reiley, and A. Sadrieh (2007). Online auctions. In T. Hendershott (Ed.), *Handbook of Information Systems and Economics*, pp. 571–628. Elsevier Science.

Riley, J. and W. Samuelson (1981). Optimal auctions. *American Economic Review 71*, 381–392.

Selten, R. (1975). A reexamination of the perfectness concept for equilibrium. *International Journal of Game Theory 4*, 25–55.

Selten, R. (1983). Evolutionary stability in extensive two-person games. *Mathematical Social Sciences 5*, 269–363.

Skreta, V. (2013). Optimal auction design under non-commitment. *working paper*.

Vartiainen, H. (2013). Auction design without commitment. *Journal of the European Economic Association 11*, 316–342.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance 16*, 8–37.